1st degree $x = a$,
2nd " $x = \frac{1}{2}a \pm \sqrt{b}$,
4th " $x = \frac{1}{4}a \pm \sqrt{b} \pm \sqrt{(c \pm d\sqrt{b})}$,
8th " $x = \frac{1}{8}a \pm \sqrt{b} \pm \sqrt{(c \pm d\sqrt{b})} \pm \sqrt{[e \pm f\sqrt{b} \pm \sqrt{(g \pm h\sqrt{b})}]}$
3rd " $x = \frac{1}{3}a + b$ or $\frac{1}{3}a - \frac{1}{2}b \pm \sqrt{c}$,
6th " $x = \frac{1}{6}a + \frac{1}{2}b \pm \sqrt{c}$ or $\frac{1}{6}a - \frac{1}{4}b \pm \sqrt{d} \pm \sqrt{(e \pm f\sqrt{d})}$.
5th " $x = \frac{1}{5}a + b$ or $\frac{1}{5}a - \frac{1}{4}b \pm \sqrt{[c \pm \sqrt{d} \pm \sqrt{(e \pm f\sqrt{d})}]}$,
10th " $x = \frac{1}{10}a +$ &c.

To solve an equation of any degree, we form an equation of the required degree from the roots as given above and equate the coefficients thus formed to those of the given equation, and we shall have n equations containing n unknown quantities. Eliminating between these, the solution of the final equation containing one unknown quantity will give the solution of the original equation. The degree of the final equation may generally be determined by *a priori* considerations. Thus, if n is odd, the degree of the final equation will be n also, and no progress has been made; but when n is any power of 2 the forms of the n roots are symmetrical and we may reasonably expect some analogy in their solutions. The solution of the biquabratic is known to depend on that of the cubic, and similarly that of the *octic* should depend upon that of the *quintic*. This would give relations of identity between the roots as follows: r, r', r'', &c., being roots of the octic and q', q'', &c., roots of the corresponding quintic,

$$r' = f_1q' = f_5q'' = f_6q''' = f_7q'''' = f_8q''''',$$
$$r'' = f_2q' = f_6q'' = f_7q''' = f_8q'''' = f_5q''''',$$
$$r''' = \text{\&c.}$$

There being 8 root forms, and the five values of q entering into each, there would be 40 different roots, unless the identities above written are true.

ON THE SUM OF THE CUBES OF ANY NUMBER OF TERMS OF ANY ARITHMETICAL SERIES.

BY L. P. SHIDY, U. S. COAST SURVEY.

THE following propositions are thought to be original, but mathematicians may be already familiar with them.

I. The sum of the cubes of any number of consecutive terms of any arithmetical series is divisible by the sum of that series.

II. In any arithmetical series whose first term is equal to the common difference, the sum of the cubes of any number of terms is equal to the product of the common difference by the square of the sum of the series.

Let a, $a+d$, $a+2d$, ... $a+(n-1)d$ be any arithmetical series, and let $S=$ the sum of n terms: then

$$S=\tfrac{1}{2}n[2a+(n-1)d]; \quad \ldots\ldots\ldots\ldots \quad (1)$$

and denoting the sum of the cubes of the terms by S', we have

$$S'=a^3+(a+d)^3+(a+2d)^3+\ldots\ldots+[a+(n-1)d]^3.$$

Expanding, and adding similar terms gives

$$S'=na^3+a^2d[3+6+9+\ldots+3(n-1)]+ad^2[3+12+27+\ldots+3(n-1)^2]$$
$$+d^3[1+8+27+\ldots+(n-1)^3]. \quad \ldots\ldots\ldots\ldots \quad (2)$$

But $3+6+9+\ldots+3(n-1)=\frac{3}{2}(n^2-n)$, $3+12+27+\ldots\ldots+3(n-1)^2$ $=3[1^2+2^2+3^2+\ldots\ldots+(n-1)^2]=[1+2+3+\ldots\ldots+(n-1)](2n-1)$ $=\frac{1}{2}(n^2-n)(2n-1)$, and $1+8+27+\ldots+(n-1)^3=1^3+2^3+3^3+\ldots$ $+(n-1)^3=[\frac{1}{2}(n^2-n)]^2$. Substituting these values in equation (2) gives,

$$S'=na^3+3a^2d[\tfrac{1}{2}(n^2-n)]+ad^2[\tfrac{1}{2}(n^2-n)](2n-1)+d^3[\tfrac{1}{2}(n^2-n)^2].$$

Performing the operations indicated this becomes,

$$S'=na^3+\tfrac{3}{2}n^2a^2d-\tfrac{3}{2}na^2d+n^3ad^2-\tfrac{3}{2}n^2ad^2+\tfrac{1}{2}nad^2+\tfrac{1}{4}n^4d^3-\tfrac{1}{2}n^3d^3+\tfrac{1}{4}n^2d^3.$$

Factoring, $S'=\frac{1}{2}n(2a+nd-d)(a^2+nad-ad+\frac{1}{2}n^2d^2-\frac{1}{2}nd^2)$,

or $$S'=\tfrac{1}{2}n[2a+(n-1)d](a^2+nad-ad+\tfrac{1}{2}n^2d^2-\tfrac{1}{2}nd^2), \ldots \quad (3)$$

in which the factor $\frac{1}{2}n[2a+(n-1)d]=S$. Therefore the sum of the cubes of the terms is divisible by the sum of the series, as announced in Prop. I.

If we square both members of equation (1) we have

$$S^2=(\tfrac{1}{2}n)^2(4a^2+4nad-4ad+n^2d-2nd+d^2),$$ which becomes $S^2=(\frac{1}{2}n)^2(n^2d^2+2nd^2+d^2)$ when $a=d$. Making $a=d$ in equation (3) it becomes $S'=(\frac{1}{2}n)^2(n^2d^2+2nd^2+d^2)d=S^2d$; which proves Prop. II.

SOLUTION OF NUMERICAL EQUATIONS OF HIGHER DEGREES WITH SECONDARY (IMAGINARY) ROOTS.

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I. ANY algebraic equation of the $2n^{\text{th}}$ degree, and having n pairs of secondary conjugate roots, will have primary (real) coefficients, and its general form will be,

$$(1) \qquad x^{2n}+A_1x^{2n-1}+\ldots\ldots+A_{2n-1}x+A_{2n}=0.$$